ECEN 227 - Introduction to Finite Automata and Discrete Mathematics

ECEN 227

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Talk Overview

1. Mathematical definitions
2. Introduction to proofs
3. Proof by Exhaustion
4. Proof by Counter Example
5. Direct Proof
6. Proof by Contrapositive
7. Indirect Proof
8. Proof by Cases

Outline

Even and Odd Integers

Even Integer

An integer x is even if there is an integer k such that $x = 2k$

Ex.

- $0 = 2*0$
- $2 = 2*1$
- $4 = 2*2$

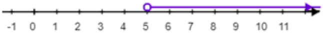
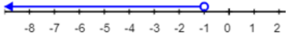
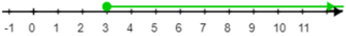
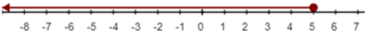
Odd Integer

An integer x is odd if there is an integer k such that $x = 2k+1$.

Ex.

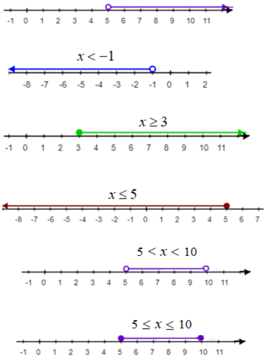
- $1 = 2*0+1$
- $3 = 2*1+1$
- $5 = 2*2+1$

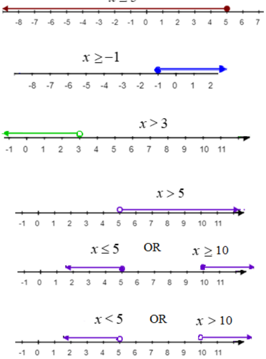
Equality and Inequality

Symbol	Words		Example
>	Greater than		$x > 5$ 
<	Less than		$x < -1$ 
≥	Greater than OR equal	at least	$x \geq 3$ 
≤	Less than OR equal	at most	$x \leq 5$ 
< <	Between (Inclusive)		$5 < x < 10$ 
≤ ≤	Between (Exclusive)		$5 \leq x \leq 10$ 

Negation of the inequalities

Symbol	Words		Example	Negation
>	Greater than		$x > 5$	$x \le 5$
<	Less than		$x < -1$	$x \ge -1$
≥	Greater than OR equal	at least	$x \ge 3$	$x > 3$
≤	Less than OR equal	at most	$x \le 5$	$x > 5$
< <	Between (Inclusive)		$5 < x < 10$	$x \le 5$ OR $x \ge 10$
≤ ≤	Between (Exclusive)		$5 \le x \le 10$	$x < 5$ OR $x > 10$





Divides

Divides

An integer x divides an integer y if and only if y = kx, for some integer k.

Ex

- 5 divides 20, in other words 20=5*4

The fact that x divides y is denoted $x \mid y$. If x does not divide y, then that fact is denoted $x \nmid y$.

If x divides y, then y is said to be a multiple of x, and x is a factor or divisor of y.

Prime and Composite Numbers

Prime Numbers

An integer n is prime if and only if n > 1, and for every positive integer m, if m divides n, then m = 1 or m = n.

Ex.

- n=7
- n=13

Combosite Numbers

An integer n is composite if and only if $n > 1$, and there is an integer m such that 1 < m < n and m divides n.

Ex.

- n=10 , m = 2 or m = 5

Outline

Introduction

Theorem

A theorem is a statement that can be proven to be true.

Axiom

It is a statement which is accepted without question, and which has no proof.

Proof

A proof is of a series of steps, each of which follows logically from assumptions, axioms, or from previously proven statements, whose final step should result in the statement or the theorem being proven.

Introduction

- One of the hardest parts of writing proofs is knowing where to start.
- Proofs have common patterns, we will cover:
  - Proof by Exhaustion.
  - Proof by Counter Example.
  - Direct Proof.
  - Proof by Contrapositive.
  - Proof by Contradiction.
  - Proof by Cases.
- Coming up with proofs requires trial and error, even for experienced mathematicians.

How to start a proof?

- Usually proofs start with One or more assumption then some statements to show the proof goal.
- Assumptions can be inferred from the theorem text.
- Goal can also be inferred from the theorem text.
- Restating the assumption and the goal is the first step in building a proof.

Example

- The average of two real numbers is less than or equal to at least one of the two numbers.

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- The difference of two odd integers is even.

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  - Assumption: Let $x = 2k+1$, $y=2j+1$

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- Among any two consecutive integers, there is an odd number and an even number.

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  - Assumption: Let x is an integer

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- The difference of two odd integers is even.
  - Assumption: Let $x = 2k+1$, $y=2j+1$
  - Goal: (x-y) is even.
- Among any two consecutive integers, there is an odd number and an even number.
  - Assumption: Let x is an integer
  - Goal: x is even and x+1 is odd or x is odd and x+1 is even

Example

Theorem

Every positive integer is less than or equal to its square.

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Proof.

- Let x be an integer $x > 0$. Name a generic object in the domain and state given assumptions about the object

□

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Proof.

- Let x be an integer $x > 0$. Name a generic object in the domain and state given assumptions about the object
- Since x is an integer and $x > 0$, then $x \geq 1$. State reasoning in complete sentence

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Every positive integer is less than or equal to its square.

Proof.

- Let x be an integer $x > 0$. Name a generic object in the domain and state given assumptions about the object
- Since x is an integer and $x > 0$, then $x \geq 1$. State reasoning in complete sentence
- Since $x > 0$, we can multiply both sides of the inequality by x to get:

$$x * x \geq 1 * x.$$

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- Since x is an integer and $x > 0$, then $x \geq 1$. State reasoning in complete sentence
- Since $x > 0$, we can multiply both sides of the inequality by x to get:

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- Simplify the expression we get

$$x^2 \geq x.$$

□

Outline

Prove by Exhaustion

- For universal statements, if the domain is small, it may be easiest to prove the statement by checking each element individually.

Theorem

for $n \in \{-1, 0, 1\}$ *we have* $n^2 = |n|$

Proof.

- $n = -1: \quad (-1)^2 = 1 = |-1|.$

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- $n = 0 : \quad (0)^2 = 0 = |0|.$

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Proof.

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- $n = 0 : \quad (0)^2 = 0 = |0|.$
- $n = 1 : \quad (1)^2 = 1 = |1|.$

□

Excercise

Proof by exhaustion

- For every integer n such that $0 \leq n < 4$, $2^{(n+2)} > 3^n$.

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- For every integer n such that $0 \leq n < 4$, $2^{(n+2)} > 3^n$.
  - When $n = 0$, $2^{(0+2)} = 4$ and $3^0 = 1$. $4 > 1$.

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- For every integer n such that $0 \leq n < 4$, $2^{(n+2)} > 3^n$.
  - When $n = 0$, $2^{(0+2)} = 4$ and $3^0 = 1$. $4 > 1$.
  - When $n = 1$, $2^{(1+2)} = 8$ and $3^1 = 3$. $8 > 3$.

Excercise

Proof by exhaustion

- For every integer n such that $0 \leq n < 4$, $2^{(n+2)} > 3^n$.
  - When $n = 0$, $2^{(0+2)} = 4$ and $3^0 = 1$. $4 > 1$.
  - When $n = 1$, $2^{(1+2)} = 8$ and $3^1 = 3$. $8 > 3$.
  - When $n = 2$, $2^{(2+2)} = 16$ and $3^2 = 9$. $16 > 9$.

Excercise

Proof by exhaustion

- For every integer n such that $0 \leq n < 4$, $2^{(n+2)} > 3^n$.
  - When $n = 0$, $2^{(0+2)} = 4$ and $3^0 = 1$. $4 > 1$.
  - When $n = 1$, $2^{(1+2)} = 8$ and $3^1 = 3$. $8 > 3$.
  - When $n = 2$, $2^{(2+2)} = 16$ and $3^2 = 9$. $16 > 9$.
  - When $n = 3$ $2^{(3+2)} = 32$ and $3^3 = 27$. $32 > 27$.

Outline

Counter example

- A counterexample is an assignment of values to variables.
- A counterexample can be used to proof/disproof a logical statement.

Ex

" If n is an integer greater than 1, then $(1.1)^n < n^{10}$ ".

For n = 686, the statement is false because

$$(1.1)^{686} > 686^{10}$$

Conditional statements proof/disproof

- A counterexample can be used to disproof a conditional statement must satisfy all the hypotheses and contradict the conclusion.
- Proofing conditional statement can use proof by exhaustion or other mathematical derivation to reach the goal.

Ex.

- **Theorem:** For any real number x, if x $\geq$ 0 and x < 1, then $x^2 < x$.

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- **Theorem:** if x is positive integer, then $1/x < x$.

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    - Counter example: x = 0, satisfy the hypotheses and contradict the conclusion
- **Theorem:** if x is positive integer, then $1/x < x$.
    - Counter example: x = 1, satisfy the hypotheses and contradict the conclusion

Universal Statement Proof/Disproof

- A counterexample can be used to disproof a universal statement.
- Proofing universal statement can use proof by exhaustion or other mathematical derivation to reach the goal.

Ex.

- **Theorem:** All primes are odd.

Universal Statement Proof/Disproof

- A counterexample can be used to disproof a universal statement.
- Proofing universal statement can use proof by exhaustion or other mathematical derivation to reach the goal.

Ex.

- **Theorem:** All primes are odd.
  - Counter example: $x = 2$, prime but not odd

Existential Statement Proof

- A counterexample can be used to proof a existential statement, this method called constructive proof of existence.

Ex.

- **Theorem:** There is an integer that can be written as the sum of the squares of two positive integers in two different ways.

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  - $50 = 1^2 + 7^2$

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- **Theorem:** There are two consecutive positive integers whose product is less than their sum.
  - 1 and 2

Existential Statement DisProof

- Disproofing existential statement can use proof by exhaustion or other mathematical derivation to reach the **negation** of the goal

Ex.

- **Theorem:** There is a real number whose square is negative.

Existential Statement DisProof

- Disproofing existential statement can use proof by exhaustion or other mathematical derivation to reach the **negation** of the goal

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- **Theorem:** There is a real number whose square is negative.
  - Disproof Goal: It is not true that there is a real number whose square is negative.

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Ex.

- **Theorem:** There is a real number whose square is negative.
  - Disproof Goal: It is not true that there is a real number whose square is negative.
  - Disproof Goal: Every real number does not have a negative square.

Existential Statement DisProof

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Ex.

- **Theorem:** There is a real number whose square is negative.
  - Disproof Goal: It is not true that there is a real number whose square is negative.
  - Disproof Goal: Every real number does not have a negative square.
  - Disproof Goal: Every real number have a square that is greater than or equal zero.

Excercise

Find a counterexample to show that each of the statements is false.

- Every month of the year has 30 or 31 days.

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  - February

Excercise

Find a counterexample to show that each of the statements is false.

- Every month of the year has 30 or 31 days.
  - February
- If n is an integer and n^2 is divisible by 4, then n is divisible by 4.

Excercise

Find a counterexample to show that each of the statements is false.

- Every month of the year has 30 or 31 days.
  - February
- If n is an integer and n^2 is divisible by 4, then n is divisible by 4.
  - $n = 2$

Excercise

Find a counterexample to show that each of the statements is false.

- Every month of the year has 30 or 31 days.
  - February
- If n is an integer and n^2 is divisible by 4, then n is divisible by 4.
  - $n = 2$
- For every positive integer x, $x^3 < 2^x$

Excercise

Find a counterexample to show that each of the statements is false.

- Every month of the year has 30 or 31 days.
  - February
- If n is an integer and n^2 is divisible by 4, then n is divisible by 4.
  - n = 2
- For every positive integer x, $x^3 < 2^x$
  - x = 3

Outline

Direct Proof

Used to proof Conditional Statements such as $p \rightarrow c$ are correct.

Direct Proof

In a direct proof of a conditional statement, the hypothesis p is assumed to be true and the conclusion c is proven as a direct result of the assumption.

Direct Proof (Example 1)

Theorem

if x is an odd integer and y is an even integer then:

$$x + y \text{ is odd}$$

Proof.

Assume:

$\because$ x = 2j+1

Direct Proof (Example 1)

Theorem

if x is an odd integer and y is an even integer then:

$$x + y \text{ is odd}$$

Proof.

Assume:

$\because$ x = 2j+1

$\because$ y = 2k

Direct Proof (Example 1)

Theorem

if x is an odd integer and y is an even integer then:

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Proof.

Assume:

$\because$ x = 2j+1

$\because$ y = 2k

Then:

$\therefore x + y$ = 2j+1+2k

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Theorem

if x is an odd integer and y is an even integer then:

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Proof.

Assume:

$\because$ x = 2j+1

$\because$ y = 2k

Then:

$\therefore x + y$ = 2j+1+2k

$\therefore x + y$ = 2(j+k)+1

Direct Proof (Example 1)

Theorem

if x is an odd integer and y is an even integer then:

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Proof.

Assume:

$\because$ x = 2j+1

$\because$ y = 2k

Then:

$\therefore x + y$ = 2j+1+2k

$\therefore x + y$ = 2(j+k)+1

$\therefore x + y$ = 2m+1 m is an integer = j+k

Direct Proof (Example 1)

Theorem

if x is an odd integer and y is an even integer then:

$$x + y \text{ is odd}$$

Proof.

Assume:

$\because$ x = 2j+1

$\because$ y = 2k

Then:

$\therefore x + y$ = 2j+1+2k

$\therefore x + y$ = 2(j+k)+1

$\therefore x + y$ = 2m+1 m is an integer = j+k

$\therefore x + y$ is odd

□

Direct Proof (Example 2)

Theorem

if r and s are rational numbers then:

$$r + s \text{ is a rational number.}$$

Proof.

Assume:

$\because r = \frac{a}{b}$ a and b are integers $b \neq 0$

Direct Proof (Example 2)

Theorem

if r and s are rational numbers then:

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$\because$ s $= \frac{c}{d}$ c and d are integers $d \neq 0$

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if r and s are rational numbers then:

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Assume:

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$\because$ s $= \frac{c}{d}$ c and d are integers $d \neq 0$

Then:

$\therefore$ r + s$= \frac{a}{b} + \frac{c}{d}$

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Then:

$\therefore$ r + s$= \frac{a}{b} + \frac{c}{d}$

$\therefore$ r + s$= \frac{(ad+cb)}{db}$

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$\therefore$ r+s $= \frac{j}{k}$ j= ad + cb and k = db are integers $k \neq 0$

Direct Proof (Example 2)

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if r and s are rational numbers then:

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Proof.

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Then:

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$\therefore$ r+s $= \frac{j}{k}$ j= ad + cb and k = db are integers $k \neq 0$

$\therefore$ r+s is rational □

Direct Proof (Example 3)

Theorem

if x and y are positive real numbers then:

$$\frac{x}{y} + \frac{y}{x} \geq 2$$

Proof.

Assume:

$\because$ x and y are real numbers

Direct Proof (Example 3)

Theorem

if x and y are positive real numbers then:

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Proof.

Assume:

$\because$ x and y are real numbers

Then:

$\therefore x - y$ is also a real number.

Direct Proof (Example 3)

Theorem

if x and y are positive real numbers then:

$$\frac{x}{y} + \frac{y}{x} \geq 2$$

Proof.

Assume:

$\because$ x and y are real numbers

Then:

$\therefore x - y$ is also a real number.

$\therefore (x - y)^2 \geq 0$, the square of any real number is greater than or equal to 0.

Direct Proof (Example 3)

Theorem

if x and y are positive real numbers then:

$$\frac{x}{y} + \frac{y}{x} \geq 2$$

Proof.

Assume:

$\because$ x and y are real numbers

Then:

$\therefore x - y$ is also a real number.

$\therefore (x - y)^2 \geq 0$, the square of any real number is greater than or equal to 0.

$\therefore x^2 - 2xy + y^2 \geq 0$

Direct Proof (Example 3)

Theorem

if x and y are positive real numbers then:

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Proof.

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$\because$ x and y are real numbers

Then:

$\therefore x - y$ is also a real number.

$\therefore (x - y)^2 \geq 0,$ the square of any real number is greater than or equal to 0.

$\therefore x^2 - 2xy + y^2 \geq 0$

$\therefore \frac{x}{y} - 2 + \frac{y}{x} \geq 0$ divide both sides of the inequality by xy

Direct Proof (Example 3)

Theorem

if x and y are positive real numbers then:

$$\frac{x}{y} + \frac{y}{x} \geq 2$$

Proof.

Assume:

$\because$ x and y are real numbers

Then:

$\therefore x - y$ is also a real number.

$\therefore (x - y)^2 \geq 0$, the square of any real number is greater than or equal to 0.

$\therefore x^2 - 2xy + y^2 \geq 0$

$\therefore \frac{x}{y} - 2 + \frac{y}{x} \geq 0$ divide both sides of the inequality by xy

$\therefore \frac{x}{y} + \frac{y}{x} \geq 2$ Adding 2 to both sides

$\square$

Outline

Proof by Contrapositive

- Used to proof Conditional Statements such as $p \rightarrow c$ are correct.
- Remember if $p \rightarrow c$ then $\neg c \rightarrow \neg p$ (i.e., contrapositive)

Proof by Contrapositive

In a proof by contrapositive of a conditional statement, the conclusion c is assumed to be false (i.e., $\neg c = true$) and the hypothesis p is proven as false (i.e., $\neg p = true$).

Proof by Contrapositive (Example 1)

Theorem

If $3n + 7$ is an odd integer, then n is an even integer

Proof.

Assume:

n is an odd integer   negation of conclusion

Proof by Contrapositive (Example 1)

Theorem

If $3n + 7$ is an odd integer, then n is an even integer

Proof.

Assume:

n is an odd integer negation of conclusion

Then:

$\because$ $n = 2k + 1$ for some integer k

Proof by Contrapositive (Example 1)

Theorem

If $3n + 7$ is an odd integer, then n is an even integer

Proof.

Assume:

n is an odd integer negation of conclusion

Then:

$\because n = 2k + 1$ for some integer k

$\therefore 3n + 7 = 3(2k + 1) + 7$

Proof by Contrapositive (Example 1)

Theorem

If $3n + 7$ is an odd integer, then n is an even integer

Proof.

Assume:

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$\therefore 3n + 7 = 6k + 3 + 7$

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$\therefore 3n + 7 = 6k + 10$

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$\therefore 3n + 7 = 2(3k + 5)$

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$\therefore 3n + 7 = 2\ m$

Therefore: $3n + 7$ is an even integer.

□

Proof by Contrapositive (Example 2)

Theorem

For every integer x, if x^2 is even, then x is even.

Proof.

Assume:

x is an odd integer — negation of conclusion

Proof by Contrapositive (Example 2)

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Proof.

Assume:

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Then:

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For every integer x, if x^2 is even, then x is even.

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Assume:

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Then:

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$\therefore x^2 = (2k+1)^2$

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For every integer x, if x^2 is even, then x is even.

Proof.

Assume:

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Then:

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$\therefore x^2 = (2k+1)^2$

$\therefore x^2 = 4k^2 + 4k + 1$

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$\therefore x^2 = 2(2k^2 + 2k) + 1$

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$\therefore x^2$ is odd   negation of hypothesis

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Proof by Contrapositive (Example 3)

Theorem

For every positive real number r, if r is irrational, then $\sqrt{r}$ is also irrational.

Proof by Contrapositive (Example 3)

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For every positive real number r, if r is irrational, then $\sqrt{r}$ *is also irrational.*

Proof.

Assume:

$\sqrt{r}$ is rational number   negation of conclusion

Proof by Contrapositive (Example 3)

Theorem

For every positive real number r, if r is irrational, then $\sqrt{r}$ is also irrational.

Proof.

Assume:

$\sqrt{r}$ is rational number negation of conclusion

Then:

$\therefore \sqrt{r} = \frac{x}{y}$

Proof by Contrapositive (Example 3)

Theorem

For every positive real number r, if r is irrational, then $\sqrt{r}$ is also irrational.

Proof.

Assume:

$\sqrt{r}$ is rational number negation of conclusion

Then:

$\therefore \sqrt{r} = \frac{x}{y}$

$\therefore r = \frac{x^2}{y^2}$ Squaring both sides

Proof by Contrapositive (Example 3)

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For every positive real number r, if r is irrational, then $\sqrt{r}$ is also irrational.

Proof.

Assume:

$\sqrt{r}$ is rational number negation of conclusion

Then:

$\therefore \sqrt{r} = \frac{x}{y}$

$\therefore r = \frac{x^2}{y^2}$ Squaring both sides

Note : x and y are integers, also x^2 and y^2 are both integers.
Since $y \neq 0$, y^2 is also non-zero. The number r is equal to the ratio of two integers in which the denominator is non-zero.

Proof by Contrapositive (Example 3)

Theorem

For every positive real number r, if r is irrational, then $\sqrt{r}$ is also irrational.

Proof.

Assume:

$\sqrt{r}$ is rational number — negation of conclusion

Then:

$\therefore \sqrt{r} = \frac{x}{y}$

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Note : x and y are integers, also x^2 and y^2 are both integers.
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r is rational — negation of hypothesis

Proof by Contrapositive (Example 3)

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Proof.

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□

Outline

Proof by Contradiction (Indirect Proof)

Proof by contradiction

A proof by contradiction starts by assuming that the theorem is false and then shows that some logical inconsistency arises as a result of this assumption.

- Unlike direct proofs a proof by contradiction can be used to prove theorems that are not conditional statements.

Ex. To prove the statement $p \to q$ then the beginning assumption is $p \wedge \neg q$ which is logically equivalent to $\neg(p \to q)$.

Proof by Contradiction (Example 1)

Theorem

If a and b are positive real numbers then $\sqrt{a} + \sqrt{b} \neq \sqrt{a+b}$

Proof.

Assume:

1. $a > 0, b > 0$
2. $\sqrt{a} + \sqrt{b} = \sqrt{a+b}$

Proof by Contradiction (Example 1)

Theorem

If a and b are positive real numbers then $\sqrt{a} + \sqrt{b} \neq \sqrt{a+b}$

Proof.

Assume:

1. $a > 0, b > 0$
2. $\sqrt{a} + \sqrt{b} = \sqrt{a+b}$

Then:

$\therefore (\sqrt{a} + \sqrt{b})^2 = (\sqrt{a+b})^2$ Squaring both sides of 2

Proof by Contradiction (Example 1)

Theorem

If a and b are positive real numbers then $\sqrt{a} + \sqrt{b} \neq \sqrt{a+b}$

Proof.

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1. $a > 0, b > 0$
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Then:

$\therefore (\sqrt{a} + \sqrt{b})^2 = (\sqrt{a+b})^2$ Squaring both sides of 2

$\therefore (\sqrt{a}^2 + 2\sqrt{ab} + \sqrt{b}^2) = a + b$

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Then:

$\therefore (\sqrt{a} + \sqrt{b})^2 = (\sqrt{a+b})^2$ Squaring both sides of 2

$\therefore (\sqrt{a}^2 + 2\sqrt{ab} + \sqrt{b}^2) = a + b$

$\therefore (\sqrt{a}^2 + 2\sqrt{ab} + \sqrt{b}^2) = a + b$

$\therefore a + 2\sqrt{ab} + b = a + b$ Subtract a+b

Proof by Contradiction (Example 1)

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If a and b are positive real numbers then $\sqrt{a} + \sqrt{b} \neq \sqrt{a+b}$

Proof.

Assume:

1. $a > 0, b > 0$
2. $\sqrt{a} + \sqrt{b} = \sqrt{a+b}$

Then:

$\therefore (\sqrt{a} + \sqrt{b})^2 = (\sqrt{a+b})^2$ Squaring both sides of 2

$\therefore (\sqrt{a}^2 + 2\sqrt{ab} + \sqrt{b}^2) = a + b$

$\therefore (\sqrt{a}^2 + 2\sqrt{ab} + \sqrt{b}^2) = a + b$

$\therefore a + 2\sqrt{ab} + b = a + b$ Subtract a+b

$\therefore 2\sqrt{ab} = 0$

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Proof.

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2. $\sqrt{a} + \sqrt{b} = \sqrt{a+b}$

Then:

$\therefore (\sqrt{a} + \sqrt{b})^2 = (\sqrt{a+b})^2$ Squaring both sides of 2

$\therefore (\sqrt{a}^2 + 2\sqrt{ab} + \sqrt{b}^2) = a + b$

$\therefore (\sqrt{a}^2 + 2\sqrt{ab} + \sqrt{b}^2) = a + b$

$\therefore a + 2\sqrt{ab} + b = a + b$ Subtract a+b

$\therefore 2\sqrt{ab} = 0$

Either a = 0 or b = 0, Contradiction with 1

□

Proof by Contradiction (Example 2)

$\sqrt{2}/2$ is an irrational number.

Assume:
$\sqrt{2}/2$ is rational
Then:

Proof by Contradiction (Example 2)

$\sqrt{2}/2$ is an irrational number.

Assume:
$\sqrt{2}/2$ is rational
Then:
$\therefore \sqrt{2}/2 = \frac{a}{b}$ a and b are integers $b \neq 0$

Proof by Contradiction (Example 2)

$\sqrt{2}/2$ is an irrational number.

Assume:
$\sqrt{2}/2$ is rational
Then:
$\therefore \sqrt{2}/2 = \frac{a}{b}$ a and b are integers $b \neq 0$
$\therefore \sqrt{2} = \frac{2a}{b}$ multiplying both sides by 2

Proof by Contradiction (Example 2)

$\sqrt{2}/2$ is an irrational number.

Assume:
$\sqrt{2}/2$ is rational
Then:
$\therefore \sqrt{2}/2 = \frac{a}{b}$ a and b are integers $b \neq 0$
$\therefore \sqrt{2} = \frac{2a}{b}$ multiplying both sides by 2
$\therefore \sqrt{2} = \frac{c}{b}$ where both c and b are integers

Proof by Contradiction (Example 2)

$\sqrt{2}/2$ is an irrational number.

Assume:
$\sqrt{2}/2$ is rational
Then:
$\therefore \sqrt{2}/2 = \frac{a}{b}$ a and b are integers $b \neq 0$
$\therefore \sqrt{2} = \frac{2a}{b}$ multiplying both sides by 2
$\therefore \sqrt{2} = \frac{c}{b}$ where both c and b are integers
$\therefore \sqrt{2}$ is rational which contradicts that $\sqrt{2}$ is irrational number.

Proof by Contradiction (Example 3)

Theorem

Among any group of 25 people, there must be at least three who are all born in the same month.

Proof by Contradiction (Example 3)

Theorem

p: group of 25 people,

q: there must be at least three who are all born in the same month.

$p \rightarrow q$

Proof by Contradiction (Example 3)

Theorem

- x_1 : *# of people in Jan*
- x_2 : *# of people in Feb*
- $\ldots$
- x_{12} : *# of people in Dec*
- $x_1 + x_2 + \cdots + x_{12} = 25$
- $(x_1 + x_2 + \cdots + x_{12} = 25) \rightarrow ((x_1 \geq 3) \vee \ldots \vee (x_{12} \geq 3))$

Proof by Contradiction (Example 3)

Proof.

Assume:

1. $(x_1 + x_2 + \cdots + x_{12} = 25)$
2. $((x_1 \leq 2) \wedge \ldots \wedge (x_{12} \leq 2))$

Then.

$\therefore (x_1 + x_2 + \cdots + x_{12}) \leq (2 + x_2 + \cdots + x_{12})$

$\therefore (x_1 + x_2 + \cdots + x_{12}) \leq (2 + 2 + \cdots + x_{12})$

$\therefore (x_1 + x_2 + \cdots + x_{12}) \leq 24$

Contradiction with 1. □

Outline

Proof by cases

- A proof by cases of a universal statement such as $\forall x P(x)$ breaks the domain for the variable x into different cases and gives a different proof for each case.
- Every value in the domain must be included in at least one case.

Example 1

Theorem

For every integer x, $x^2 - x$ is an even integer.

Proof.

Case 1 x is even: $x = 2k$ for some integer k

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Theorem

For every integer x, $x^2 - x$ is an even integer.

Proof.

Case 1 x is even: x = 2k for some integer k

$$\begin{aligned} x^2 - x &= (2k)^2 - 2k \\ &= 4k^2 - 2k \\ &= 2(2k^2 + k) \\ &= 2d \end{aligned}$$

∴ theorem is correct for Case 1 □

Example 1

Theorem

For every integer x, $x^2 - x$ is an even integer.

Proof.

Case 2 x is odd: $x = 2k + 1$ for some integer k

Example 1

Theorem

For every integer x, $x^2 - x$ is an even integer.

Proof.

Case 2 x is odd: x = 2k +1 for some integer k

$$
\begin{aligned}
x^2 - x &= (2k+1)^2 - (2k+1) \\
&= 4k^2 + 4k + 1 - (2k+1) \\
&= 4k^2 + 2k \\
&= 2(2k^2 + k) \\
&= 2d
\end{aligned}
$$

∴ theorem is correct for Case 2 □

Example 2

Theorem

For any real number x, $|x+5| - x > 1$

Proof.

Case 1. $(x+5) \geq 0$: Therefore : $|x+5| = +(x+5)$

$$\begin{aligned} |x+5| - x &= (x+5) - x \\ &= 5 > 1 \end{aligned}$$

$\therefore$ theorem is correct for Case 1 □

Questions